

Classical microscopic derivation of the relativistic hydrodynamics equations

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We present microscopic derivation of the relativistic hydrodynamics (RHD) equations directly from mechanics omitting derivation of kinetic equation. We derive continuity equation and energy-momentum conservation law. We also derive equation of evolution of particles current. In non-relativistic hydrodynamics equation of particles current evolution coincide with the equation of momentum evolution, the Maxwell's equations contain concentration and electric current (which proportional to the particles current), so, to get a close set of equations we should have equations of evolution of the concentration and the particles current. Evolution of the particles current depends on the electrical and magnetic fields. Thus, we obtain the set of the RHD equations as the set of the continuity equation, the equation of particles current and the Maxwell equations. This approximation does not require to include the evolution of momentum and allows to consider physical problems. Certainly, particles current evolution equation contains some new functions which we can express via concentration and particles current or we can derive equation for this functions, and, thus, get to more general approximation. This approximation also developed and discussed in this paper.

I. INTRODUCTION

Relativistic kinetics and hydrodynamics have been in a center of attention. A lot of papers and books dedicated to this topic have been published in last several years. Wide recent review of relativistic kinetics has been presented in R. Hakim's book [1], but we also should mention a paper about relativistic kinetics [2], where was suggested a method of equations derivation which we suppose to use for derivation of hydrodynamics equations.

When one considers the relativistic hydrodynamics equations he usually uses conservation laws: conservation of particles number and momentum and energy, in Lorentz covariant form, along with the Maxwell equations. To get a close set of the hydrodynamic equations we should present a connections between the momentum density \mathbf{P} and the velocity field \mathbf{v} . In the relativistic hydrodynamics (RHD) we have two equations instead of the Euler equation in the non-relativistic hydrodynamics, where the Euler equation is both the momentum balance equation and the equation of evolution of the particles current \mathbf{j} (which emerges in the continuity equation and proportional to the velocity field $\mathbf{j} = n\mathbf{v}$, where n is the concentration of particles). In the RHD's these equations are different. We can use particles current evolution instead of evolution of momentum density. So, we do not need to find connection between \mathbf{P} and $\mathbf{v} = \mathbf{j}/n$. In the Newton's mechanics and it's relativistic generalization the law of momentum evolution is the basic dynamical law. It gives us microscopic dynamical picture. When we are going to derive macroscopic dynamical equations we should find one containing information keeping in the Newton's second law, and the particles current evolution equation is one of them, as well as the momentum balance equation, these two equations match in the non-relativistic case.

When one has relativistic momentum balance equation it is a hard job to find a connection between momentum

density $\mathbf{P}(\mathbf{r}, t)$ and particles current $\mathbf{j}(\mathbf{r}, t) = n(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)$ to obtain the close set of the hydrodynamics equations for the system of particles with the temperature. Where is the connection between the momentum $\mathbf{p}_i(t)$ and the velocity $\mathbf{v}_i(t)$ of the one particle

$$\mathbf{p}_i(t) = \frac{m_i \mathbf{v}_i(t)}{\sqrt{1 - \frac{v_i^2(t)}{c^2}}}, \quad (1)$$

and

$$\mathbf{v}_i(t) = c \frac{\mathbf{p}_i(t)}{p_{0i}(t)}, \quad (2)$$

where c is the speed of light, and p_i^0 is proportional to the energy of the particle

$$p_i^0(t)c = E_i(t) = \sqrt{p_i^2(t)c^2 + m_i^2 c^4}. \quad (3)$$

If we consider system of particles, it is no simple matter to suggest a connection between $\mathbf{P} = \sum_i m \mathbf{v}_i(t) / \sqrt{1 - v_i^2(t)/c^2}$ and $\mathbf{v} = \sum_i \mathbf{v}_i(t)$.

When we consider the relativistic dynamics the Newton's equation still can be used, but we must consider the relativistic connection (1) between $\mathbf{p}_i(t)$ and $\mathbf{v}_i(t)$, so

$$\frac{d}{dt} \mathbf{p}_i(t) = e \mathbf{E}(\mathbf{r}_i(t), t) + \frac{e}{c} \mathbf{v}_i(t) \times \mathbf{B}(\mathbf{r}_i(t), t), \quad (4)$$

where $\mathbf{E}(\mathbf{r}_i(t), t)$ and $\mathbf{B}(\mathbf{r}_i(t), t)$ are the electric and magnetic fields acting on the i -th particle. Otherwise, for particle's acceleration we have

$$\begin{aligned} \frac{d}{dt} \mathbf{v}_i(t) &= \frac{e}{m} \sqrt{1 - \frac{v_i^2(t)}{c^2}} \times \\ &\times \left(\mathbf{E}(\mathbf{r}_i(t), t) + \frac{1}{c} [\mathbf{v}_i(t), \mathbf{B}(\mathbf{r}_i(t), t)] \right. \\ &\left. - \frac{1}{c^2} \mathbf{v}_i(t) (\mathbf{v}_i(t) \mathbf{E}(\mathbf{r}_i(t), t)) \right). \end{aligned} \quad (5)$$

In this paper we pay attention to the classic mechanics and hydrodynamics, but where are also papers dedicated to the quantum hydrodynamics [3], [4], [5], and the relativistic quantum hydrodynamics [6], [7].

This paper is organized as follows. In Sec. II we present a brief review of the relativistic hydrodynamics for particles system with the small thermal velocity spread. In Sec. III we consider existing in the literature methods of obtaining of the RHD equations for the system of relativistic particles with the temperature. In Sec. IV we derive the RHD equations from the picture of microscopic motion of particles described by the Newton's law. In Sec. V we consider temporal evolution of new function appeared at derivation described in the Sec. IV. In Sec. VI we derive the equation of evolution for the energy-momentum density to show that our treatment give the well-known equation. In Sec. VII we discuss ways to obtain a close set of equations. In Sec. VIII we consider dispersion of waves in the relativistic plasma. In Sec. IX we present the brief summary of our results.

II. RELATIVISTIC HYDRODYNAMICS OF PARTICLES SYSTEM AT SMALL TEMPERATURE

As a simple example of relativistic many particle system we consider a relativistic electron beam. If it's beam is monoenergetic or it has small thermal spread of particle velocities when all particles have (near) equal velocities \mathbf{u} , so momentum of the system $\mathbf{P} = \sum_{i=1}^N m\mathbf{u}/\sqrt{1-u^2/c^2} = Nm\mathbf{u}/\sqrt{1-u^2/c^2}$ is proportional to the momentum of single particle, and it connects with the particle velocity by well-known relativistic formula. Here and below, we use \mathbf{u} as velocity field for approximately monoenergetic system of particles. Here and below, for simplicity, we consider one species.

Following by Ref.s [8], [9], [10], [11], we can write an example of using in literature set of the RHE

$$\partial_t n + \nabla(n\mathbf{v}) = 0, \quad (6)$$

and

$$\partial_t \mathbf{p} + (\mathbf{v}\nabla)\mathbf{p} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}, \quad (7)$$

where momentum of medium is presented in the form $\mathbf{p} = \gamma m\mathbf{v}$, where $\gamma = (1 - v^2/c^2)^{-1/2}$, analogously to single particle, to close the set of equations. In equations (6) and (7) n is the concentration of particles, \mathbf{v} is the velocity field, $\mathbf{p} = \mathbf{P}/n$, where \mathbf{P} is the density of momentum, \mathbf{E} and \mathbf{B} are the electric and magnetic fields.

III. SHORT REVIEW OF BASIC POINTS OF RELATIVISTIC HYDRODYNAMICS

In well-known course of theoretical physics written by L. D. Landau and E. M. Lifshitz [12], [13] we can find

a macroscopic derivation of the relativistic hydrodynamics for the system of neutral particles. This derivation based on treatment of the macroscopic bit of medium. They start their consideration from definition of the stress-energy tensor \hat{T} in the rest frame. In this case they get

$$\hat{T} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad (8)$$

where e is the inner energy density, p is the pressure, these quantities are defined in the rest frame, i.e. the frame (may be local) where is no macroscopic motion of the medium. Having (8) they use the Lorentz transformation to get in an arbitrary inertial frame. In the result they find

$$T^{ab} = hu^a u^b / c^2 + pg^{ab}, \quad (9)$$

where g^{ab} is the metric sign convention such that $g^{ab} = \text{diag}(1, -1, -1, -1)$, and $h = e + p$ is the enthalpy density, and Latin indexes $a, b = 0, 1, 2, 3$.

The stress-energy tensor contains information about density of dynamical quantities describing the system, consequently, its can be used to write equation evolution of the medium which is an analog of the non-relativistic Euler equation, and we also should write the continuity equation which we present here in the terms of the four-dimensional variables

$$\frac{\partial j^a}{\partial x^a} = 0 \quad (10)$$

is the number of particles conservation law, where j^a is the four-current.

$$\frac{\partial T^{ab}}{\partial x^b} = 0 \quad (11)$$

is the stress-energy conservation law. In formulas (10) and (11) $x^a = (ct, x, y, z)$, where t, x, y, z are the independent variables, whereas in the microscopic mechanics we have $(\mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t), t)$.

To write (8) we should consider a local frame where this piece of medium does not move. Generally speaking this piece can be involved in complex motion, and moving with acceleration relatively to some global inertial frame. Moreover, if the piece is not moving macroscopically in chosen inertial frame, its particles in their thermal motion can move with relativistic velocities. Thus, we can not use non-relativistic formulas in this frame too. Presented in the next section description is free of these restrictions.

More general approximation in compare with the described above was developed by S. M. Mahajan in 2003 [14], and effectively used in many papers, for example we present several of them [15], [16], [17]. This approximation allows to get "minimal coupling" for the system of relativistic charged particles. It includes thermal distribution of particles and founds on the Lorentz invariance of equation of motion. When we consider system of charged

particles, in right-hand side of the equation (11) the electromagnetic field emerges. Therefore, instead of (11) we have

$$\frac{\partial T^{ab}}{\partial x^b} = F^{ab} j_b, \quad (12)$$

where $F^{ab} = \partial^a A^b - \partial^b A^a$ is the tensor of the electromagnetic field and A^a is the four-potential of the electromagnetic field. Introducing $\mathbf{p} - (e/c)\mathbf{A}$ instead of \mathbf{p} we can rewrite (12) as

$$\frac{\partial \mathbf{T}^{ab}}{\partial x^b} = 0, \quad (13)$$

where \mathbf{T}^{ab} is the sum of the stress-energy tensor T^{ab} and the electromagnetic stress-energy tensor T_{EM}^{ab} , where

$$T_{EM}^{ab} = \frac{1}{4\pi} \left(F^{ac} g_{cd} F^{bd} - \frac{1}{4} g^{ab} F^{cd} F_{cd} \right)$$

In some cases, see for example sect. II of this paper, we can present T^{ab} via four-velocity $U^a = (\gamma, \gamma \mathbf{u}/c)$ as

$$T^{ab} = \partial^a U^b - \partial^b U^a. \quad (14)$$

To account the statistical information of the particles system in this formula in Ref. [14] a new tensor $S^{ab} = \partial^a (f U^b) - \partial^b (f U^a)$ was introduced. It contains parameter f , which is the function of the temperature T . Thus, in Ref. [14] a following equation was suggested

$$U_b M^{ab} = 0, \quad (15)$$

where M^{ab} is the tensor that couples the electromagnetic and the fluid fields, $M^{ab} = F^{ab} + (mc^2/e) S^{ab}$.

Following by the Ref.s [16] we can represent the space-like components of equation (15)

$$(\partial_t + \mathbf{u} \nabla)(f \gamma \mathbf{u}) = \frac{e}{m} (\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) - \frac{1}{mn\gamma} \nabla p. \quad (16)$$

We also represent corresponding continuity equation

$$\partial_t(\gamma n) + \nabla(\gamma n \mathbf{u}) = 0. \quad (17)$$

Usually equation (17) considered as the continuity equation in the arbitrary frame. From this point of view equation (6) is the continuity equation in rest frame. However, below we show that at microscopic derivation of the continuity equation in arbitrary inertial frame we get (6).

Let us repeat the problem we discuss in the paper: we want to find the form of the RHD equations for the system of particles with the temperature, so, where there is distribution of particles by velocities, and, in general, particles can move with relativistic thermal velocities. One of the ways to solve with problem we just describe following to the Ref. [14], but where was suggested the minimal coupling model. It includes the contribution of the thermal motion in the RHD's, and it does not change number of equations (and variables) in the model. However, it is interesting to see a way of widening of the model, a way of it's

further development beyond of the minimal coupling. Besides, one more crucial moment, we do not want to make connection between the density of the momentum $\mathbf{P}(\mathbf{r}, t)$ and the velocity field $\mathbf{v}(\mathbf{r}, t)$ for the system of relativistic particles with the temperature, but we want to be able to solve described problem.

Another way to find equation evolution of the four-momentum is a use of a kinetic equation.

In the physical kinetics the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ is defined in the six dimensional space of possible value of the coordinate \mathbf{r} and momentum \mathbf{p} of individual particles. An individual particle's momentum and velocity connect by formula (1). Introducing the distribution function as number of particles in the vicinity of a point \mathbf{r} of the physical space and having momentum in the vicinity of a point \mathbf{p} of the momentum space, following to the Ref. [2], we can write

$$f(\mathbf{r}, \mathbf{p}, t) =$$

$$\int_{\Delta_r} d\xi \int_{\Delta_p} d\eta \sum_{i=1}^N \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \cdot \delta(\mathbf{p} + \eta - \mathbf{p}_i(t)).$$

From this formula we see that \mathbf{p} connected with the momentum of individual particles $\mathbf{p}_i(t)$. Thus, in the kinetic equation we can write $\mathbf{p} = \gamma m \mathbf{v}$. It is well-known that the hydrodynamic functions can be found using the distribution function, for example:

$$n(\mathbf{r}, t) = \int dp f(\mathbf{r}, \mathbf{p}, t),$$

$$\mathbf{j}(\mathbf{r}, t) = \int dp \frac{\mathbf{p}}{p_0} f(\mathbf{r}, \mathbf{p}, t).$$

In this case, one should use the kinetic equation to derive the hydrodynamic equations (see for example Ref. [18]).

IV. MICROSCOPIC DERIVATION OF RELATIVISTIC HYDRODYNAMICS EQUATIONS FOR CHARGED PARTICLES

To start the derivation of the RHD equations we choose the inertial frame. Next, we consider a sphere around each point of space. Each moment of time we can calculate a number of particles (or total mass of particles) in each sphere.

To find the concentration of particles in the vicinity of a point of the three dimensional physical space we should count the number of the particles and divide it on the volume of the vicinity

$$\rho(\mathbf{r}, t) = \frac{1}{\Delta} \sum_{i=1}^{N(\mathbf{r}, t)} m_i. \quad (18)$$

Microscopic number of particles (total mass) in the vicinity of the point \mathbf{r} changes during the time. This sum also

changes from one point of space to another. It is not suitable to work with the sum which up limit of summation depend on \mathbf{r} and t . Using the Dirac's delta function we can rewrite $\rho(\mathbf{r}, t)$ in the following way

$$\rho(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N m_i \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \quad (19)$$

If we consider system of particles with equal masses we can write

$$\rho(\mathbf{r}, t) = mn(\mathbf{r}, t),$$

where $n(\mathbf{r}, t)$ is the concentration of particles.

Differentiating the particles concentration with respect to time we obtain the continuity equation

$$\partial_t n + \nabla \mathbf{j} = 0. \quad (20)$$

We do not chose any particular inertial frame. One might consider the rest frame, but we start from the microscopic description, and the notion "rest frame" does not clear from this point of view. Separate particles have no information about: do they part of the macroscopically motionless system or not? This question might be answered on macroscopic scale only!

Current of particles appears in continuity equation, it's evident form is

$$\begin{aligned} \mathbf{j}(\mathbf{r}, t) &= \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \mathbf{v}_i \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \\ &= \frac{1}{\Delta} \int d\xi \sum_{i=1}^N c \frac{\mathbf{p}_i}{p_{0i}} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \end{aligned} \quad (21)$$

For the following it is suitable to introduce the velocity field

$$\mathbf{v}(\mathbf{r}, t) = \frac{\mathbf{j}(\mathbf{r}, t)}{n(\mathbf{r}, t)}. \quad (22)$$

Differentiation quantity (21) with respect to time we get simple, but very important equation: equation of particles current evolution. One has very interesting form

$$\partial_t j^\alpha + \partial^\beta \Pi^{\alpha\beta} = e\eta E^\alpha + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} \eta^\beta B^\gamma - \frac{e}{c} \eta^{\alpha\beta} E^\beta, \quad (23)$$

where Greek indexes α, β are used for 1, 2, 3; $\Pi^{\alpha\beta}$ is the current of \mathbf{j} or the current of particles current

$$\Pi^{\alpha\beta}(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N v_i^\alpha v_i^\beta \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \quad (24)$$

In the non-relativistic case $\Pi^{\alpha\beta}$ coincides with the current of momentum.

In the equation of particles current evolution (23) three new function appear. Evident form of them are

$$\eta(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \frac{c}{p_i^0} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \quad (25)$$

$$\eta^\alpha(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \frac{c v_i^\alpha}{p_i^0} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \quad (26)$$

and

$$\eta^{\alpha\beta}(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \frac{c v_i^\alpha v_i^\beta}{p_i^0} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \quad (27)$$

In the non-relativistic limit we find that $\eta \rightarrow n/m$, $\eta^\alpha \rightarrow j^\alpha/m$, and $\eta^{\alpha\beta} \rightarrow \Pi^{\alpha\beta}/(mc)$. Comparing the first and the third terms in the right-hand side of the equation (23), including the evident form of the concentration n and $\Pi^{\alpha\beta}$, we can see that the third term is proportional to v^2/c^2 , and exists in the semi-relativistic approximation only. So, we should neglect it in the non-relativistic limit, in compare with the first term. In the result we have the usual non-relativistic Euler equation

$$m(\partial_t j^\alpha + \partial^\beta \Pi^{\alpha\beta}) = enE^\alpha + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} j^\beta B^\gamma.$$

We make our calculations in some inertial frame, but we do not consider transition from one frame to another. For such transition we can use the Lorentz transformation for the microscopic quantities, and using it we can find lows of transformation of macroscopic quantities, but this topic lay of the paper.

Equation (23) and other dynamical equations are written in the self-consistent field approximation, and we also neglect by contribution of the electric dipole and other moments of the medium. To do it we suppose that

$$E_i^\alpha(\mathbf{r}_i, t) = E_i^\alpha(\mathbf{r} + \xi, t) \approx E_i^\alpha(\mathbf{r}, t),$$

and

$$B_i^\alpha(\mathbf{r}_i, t) = B_i^\alpha(\mathbf{r} + \xi, t) \approx B_i^\alpha(\mathbf{r}, t).$$

A way of appearing of them is described in Ref.s [19], [20] in the non-relativistic case. To make the paper not to overload we left this topic for next one.

The electric \mathbf{E} and the magnetic \mathbf{B} fields arising in equation (23) (and in other dynamical equations presented in this paper) satisfy to the Maxwell's equations

$$\nabla \mathbf{B} = 0, \quad (28)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} + \frac{4\pi en \mathbf{v}}{c}, \quad (29)$$

$$\nabla \mathbf{E} = 4\pi en, \quad (30)$$

and

$$\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B}. \quad (31)$$

V. EVOLUTION OF THE NEW FUNCTIONS

Below we derive the equations of the momentum and the energy evolution, but strictly speaking we do not need them to work with the RHD. In fact we already have "equation of motion"-it is equation (23).

Equation of the current evolution (23) introduce to us the three new functions (η , η^α , and $\eta^{\alpha\beta}$) and we should find a way to close the set of the hydrodynamics equations to solve particular problems. One of possible ways to do it is to find equations of evolution of η , η^α , and $\eta^{\alpha\beta}$. Certainly we can expect that in this case some new function will appear.

We have evident form of η and η^α , so, we can differentiate them with respect to time and obtain equations of these quantities evolution. In the result we have

$$\partial_t \eta + \partial_\alpha \eta^\alpha = -e \zeta^\alpha E^\alpha, \quad (32)$$

and

$$\partial_t \eta^\alpha + \partial_\beta \eta^{\alpha\beta} = e \zeta E^\alpha + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} \zeta^\beta B^\gamma - 2 \frac{e}{c} \zeta^{\alpha\beta} E^\beta. \quad (33)$$

As we expected several new functions appear. They evident form is presented in this section below. But, from these equations (32) and (33) we find connection between η , η^α , and $\eta^{\alpha\beta}$. From (32) we see that η^α is the current of η , and, equation (33) shows us that $\eta^{\alpha\beta}$ is the current of η^α .

In general, the current of a quantity f , which we designate as f^α , might be presented in the form of

$$f^\alpha = f \cdot v^\alpha, \quad (34)$$

where v^α is the velocity field introduced above. However, we neglect by the thermal motion in this formula, but it is very useful approximation. If we include the thermal motion in the formula (34) it assume the form $f^\alpha = f \cdot v^\alpha + f^\alpha$, where f^α present contribution of the thermal motion. For example, we consider the non-relativistic current of the momentum $\Pi^{\alpha\beta}$, which coincides with the current of particle current. In this case $\Pi^{\alpha\beta} \simeq j^\alpha v^\beta$, and knowing that $j^\alpha = n v^\alpha$, we have $\Pi^{\alpha\beta} \simeq n v^\alpha v^\beta$. Including contribution of the thermal motion we can write $\Pi^{\alpha\beta} \simeq n v^\alpha v^\beta + p^{\alpha\beta}$, where $p^{\alpha\beta}$ is the tensor of pressure. Usually one consider the scalar pressure p , which connects with the tensor in following form $p^{\alpha\beta} = p \delta^{\alpha\beta}$, where $\delta^{\alpha\beta}$ is the Kronecker delta.

Thus, we can approximately write $\eta^\alpha = \eta v^\alpha$ and $\eta^{\alpha\beta} = \eta^\alpha v^\beta / c = \eta v^\alpha v^\beta / c$. In the result, to understand meaning of η , η^α , and $\eta^{\alpha\beta}$ we need to understand meaning of η only. To do it we should consider its semi-relativistic approximation.

Here, we present evident form of three functions which arise in equations (32) and (33)

$$\zeta(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \left(\frac{c}{p_i^0} \right)^2 \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \quad (35)$$

$$\zeta^\alpha(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N v_i^\alpha \left(\frac{c}{p_i^0} \right)^2 \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \quad (36)$$

and

$$\zeta^{\alpha\beta}(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N v_i^\alpha v_i^\beta \left(\frac{c}{p_i^0} \right)^2 \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \quad (37)$$

It can be shown that ζ , ζ^α , and $\zeta^{\alpha\beta}$ connect with each other in the same way as η , η^α , and $\eta^{\alpha\beta}$. Thus we have $\zeta^\alpha = \zeta v^\alpha$ and $\zeta^{\alpha\beta} = \zeta v^\alpha v^\beta / c$.

VI. STRESS-ENERGY TENSOR

Usually, at the RHD's description one consider tensor of energy-momentum. It appears at consideration of evolution of the energy-momentum four-vector density. To show our description coincides with well-known we derive the equation of the energy-momentum four-vector density evolution. The stress-energy tensor appears with.

Let's introduce the density of the momentum

$$\begin{aligned} \mathbf{P}(\mathbf{r}, t) &= \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \mathbf{p}_i \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \\ &= \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \frac{m_i \mathbf{v}_i}{\sqrt{1 - \frac{v_i^2}{c^2}}} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \end{aligned} \quad (38)$$

This quantity is a part of the stress-energy tensor T^{ab} . Their connection is

$$\mathbf{T}^{0\alpha} = c \mathbf{P}.$$

Time part of the four-momentum vector is the energy. So, we present the density of the energy

$$T^{00}(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N c \cdot p_i^0 \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \quad (39)$$

As in the cases described above we should differentiate the definition of a quantity to find an equation of it's evolution. So, we present the equation of the momentum density evolution

$$\frac{1}{c} \partial_t T^{0\alpha} + \partial^\beta T^{\alpha\beta} = e n E^\alpha + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} j^\beta B^\gamma. \quad (40)$$

In this equation only one "new" quantity appears, it is the tensor of momentum current

$$T^{\alpha\beta}(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^N \frac{c p_i^\alpha p_i^\beta}{p_i^0} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \quad (41)$$

In the same way we find the energy density evolution equation

$$\frac{1}{c} \partial_t T^{00} + \partial^\alpha T^{\alpha 0} = \frac{e}{c} \mathbf{j} \mathbf{E}, \quad (42)$$

where $T^{\alpha 0}$ is the current of energy, which simply connects with the momentum density

$$T^{\alpha 0} = T^{0\alpha}.$$

Analogously to the non-relativistic case we can write $T^{\alpha\beta} = P^\alpha v^\beta + p^{\alpha\beta}$, where $p^{\alpha\beta} \simeq p\delta^{\alpha\beta}$ is the pressure tensor and p is the isotropic pressure. Thus we can rewrite equation (40) in the following form

$$\partial_t P^\alpha + \partial^\beta (P^\alpha v^\beta + p\delta^{\alpha\beta}) = enE^\alpha + \frac{e}{c}\varepsilon^{\alpha\beta\gamma} j^\beta B^\gamma. \quad (43)$$

Evolution of the energy density T^{00} and the momentum density \mathbf{P} depends on the concentration n , the particles current \mathbf{j} , the electric \mathbf{E} and the magnetic \mathbf{B} fields. However, evolution of n , \mathbf{j} , \mathbf{E} , and \mathbf{B} do not depend on \mathbf{P} and T^{00} . Therefore, to have the closed set of the RHD equations we do not need to consider \mathbf{P} and T^{00} evolution, and put our attention for n , \mathbf{j} and η .

If we put $p = 0$ and $\mathbf{P} = n\mathbf{p}$ in equation (43) and include equation (6) or (20) (which is the same) we find that equation (43) mach with the (7).

At studying of the relativistic plasma an approximation is used for the pressure p . In order to close the set of the RHD equation one use the equation of state for an ideal gas: $p = nk_B T$, where k_B is the Boltzmann constant and T is the temperature. It we consider evident form of $\Pi^{\alpha\beta}$ and $T^{\alpha\beta}$ we will see that they have the same tensor structure, i.e. they both depend on $v_i^\alpha v_i^\beta$. Thus, we can use for $\pi^{\alpha\beta}$, where $\Pi^{\alpha\beta} = nv^\alpha v^\beta + \pi^{\alpha\beta}$, the same approximation as for $p^{\alpha\beta}$. Consequently, we can put $\pi^{\alpha\beta} = \phi \cdot \delta^{\alpha\beta}$ and $\phi = \phi(n, T)$.

VII. CLOSING OF SET OF THE HYDRODYNAMICS EQUATIONS

During the paper we attain that we can describe relativistic plasma by means of n , j^α , η^α , E^α , and B^α . In this case we include some of effects caused by temperature, but some of them we lost. To study relativistic plasma with the large temperatures we should consider contribution of thermal motion at least in η^α and $\eta^{\alpha\beta}$. However, we suppose not to consider it in the paper. We already introduced new quantity η , and we should understand it's meaning. For this purpose we consider semi-relativistic limit for η , find out it's contribution in dispersion of waves and try to approximately express it via n and η . Even after getting approximate connection between n , \mathbf{v} , and η , we suppose to consider η as an independent variable, along with concentration n and velocity field \mathbf{v} .

Thus, the semi-relativistic approximation of η has form $\eta = n/m - \Pi^{\alpha\alpha}/2mc^2$, where $\Pi^{\alpha\alpha}$ is the trace of the tensor of the current of the particles current (24), $\Pi^{\alpha\alpha} = nv^2 + 3\phi$, where ϕ is the current of the particles current on the thermal velocities. We have got it using the formula

$$\frac{1}{p_{0i}} = \frac{1}{m_i c} \left(1 - \frac{v^2}{2c^2} \right),$$

which is the semi-relativistic approximation for the inverse time component of the four-momentum.

VIII. DISPERSION OF LONGITUDINAL WAVES IN RELATIVISTIC PLASMA

We suggest that in equilibrium state the relativistic electron plasma (we suppose that ions motionless) is described by following parameters n_0 , η_0 , and $\mathbf{v}_0 = 0$. To study dynamics of small perturbation we use equations (20), (23), and (30). For the first step we suggest that ϕ depends on concentration n only, but below we will account that ϕ depends on n and η . We notice that in non-relativistic limit ϕ becomes pressure p , and η becomes concentration n , and dependence on n and η reduces to dependence on n . Considering evolution of small perturbations around the equilibrium state we can find it's dispersion dependencies, which has form of

$$\omega^2 = 4\pi e^2 \eta_0 + \left(\frac{\partial \phi}{\partial n} \right)_0 k^2, \quad (44)$$

we can see that η_0 appears instead of the equilibrium concentration n_0 , and ϕ emerges instead of the pressure p . In the semi-relativistic limit from (44) we get

$$\omega^2 = \frac{4\pi e^2 n_0}{m} - \frac{6\pi e^2 \phi_0}{mc^2} + \left(\frac{\partial \phi}{\partial n} \right)_0 k^2,$$

where we use that $\Pi_0^{\alpha\alpha} \simeq 0 + 3\phi_0$, since $\mathbf{v}_0 = 0$. Consequently, in the non-relativistic limit we have

$$\omega^2 = \frac{4\pi e^2 n_0}{m} + 3 \frac{k_B T}{m} k^2, \quad (45)$$

in the second term the equation of state of the ideal gas was used at the adiabatic condition with the rate of adiabat equals 3.

Including that ϕ depends on both concentration n and η we still get the formula (44). However if we consider the wave of particle concentration in an electron beam we obtain dispersion equation

$$(\omega - kU)^2 - \frac{1}{m} k^2 \partial_n \phi + \frac{4\pi e^2 \eta_0^2 U k}{mn_0(\omega - kU)} \partial_\eta \phi - 4\pi e^2 \eta_0 = 0, \quad (46)$$

which describes the longitudinal waves in the electron beam. It is the equation of the third degree in contrast to (44) which is equation of the second degree.

Thus we can see that an account of $\phi(\eta)$ dependence could lead to some new effects. However, to get solution we should present equation of state $\phi = \phi(n, \eta, T)$, which has not found yet. Presented equations and it's consequence give us a lot of new open questions, but they also give another view on the development of the RHD.

IX. CONCLUSION

We have presented microscopic derivation of the relativistic hydrodynamics equation. We have derive as well as well-known and new equations. Among well-known equation we can mention continuity equation, momentum balance equation and energy balance equation. We have presented the particles current evolution equation, since the particles current simply related with the velocity field $\mathbf{j} = n\mathbf{v}$. At this equation derivation new functions have appeared. To understand their meaning and to consider their influence on particles dynamics we have derive equations of theirs evolution. We have suggested the closed set of the RHD equations and used its to consider dispersion of the collective excitations in the relativistic plasma.

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